Indian Statistical Institute
Semestral Examination
Algebra IV - BMath II year.

Max Marks: 100 Time: 3 hours.

Answer **question 1**, and **any** 4 from the rest. Throughout, F denotes a field, and $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{F}_p$ denotes the field with p elements.

(1) State true or false. Justify your answers with proper reasons.
(a) Every simple extension is algebraic.
(b) Let $F \subseteq K \subseteq L$ be field extensions such that K/F and L/K are Galois extensions, then L/F is a Galois extension.
(c) The Galois group of $x^4 - 2$ over $\mathbb{F}_3$ is $\mathbb{Z}_2$.
(d) $\mathbb{F}_2(x, y)/\mathbb{F}_2(x^2, y^2)$ is a simple extension. [5×4]

(2) Let $f(x) \in F[x]$ be a separable polynomial of degree n.
(a) Show that the Galois group of $f(x)$ over F, can be embedded into the symmetric group S_n.
(b) Show that $f(x)$ is irreducible if and only if the Galois group of $f(x)$ over F, is a transitive subgroup of S_n. [Note that a subgroup H of S_n is called transitive if H acts transitively on the set $\{1, 2, \ldots, n\}$]. [$5 + 15$]

(3) (a) Let K/F be a finite extension. Prove that K is a simple extension of F if and only if there exist only finitely many subfields of K containing F.
(b) Deduce that if K/F is a finite and separable extension, then K/F is simple. [$15 + 5$]

(4) (a) Determine the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive nth root of unity.
(b) Show that if H is any subgroup of the Galois group of $\mathbb{Q}(\zeta_p)$, p a prime, then the fixed field of H is a simple extension of $\mathbb{Q}$, with primitive element $\alpha = \sum_{\sigma \in H} \sigma(\zeta_p)$.
(c) Find all intermediate extensions of $\mathbb{Q}(\zeta_{11})$ over $\mathbb{Q}$. Find the corresponding primitive elements. [$5 + 10 + 5$]

(5) (a) Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.
(b) Show that if G is any finite abelian group, then there is a subfield K of a cyclotomic field with $Gal(K/\mathbb{Q}) \cong G$. [$8 + 12$]

(6) (a) Define *solvable groups*. State a necessary and sufficient condition for a polynomial to be solvable by radicals.
(b) Is the polynomial $f(x) = x^5 - 2x^3 - 8x - 2$ over $\mathbb{Q}$ solvable by radicals? Give complete justification. [$6 + 14$]